On a Hardy's inequality

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Abstract: For fulfillment of three weighted Hardy's integral inequalities, the necessary and sufficient conditions on the weighted functions are received.

We consider the following inequality

$$\sup_{x<b}\left(u(x)\int_x^b f(s)ds\right)\le C\left\{\left(\int_J|\rho f|^p ds\right)^{1/p}+\left(\int_J v^p(x)\left(\int_x^b G(s,x)f(s)ds\right)^p dx\right)^{1/p}\right\}, \quad (1)$$

where $f\ge 0$, and the functions $u(\cdot),\rho(\cdot),v(\cdot)$ are non-negative and continuous on $J=(a,b),\ -\infty\le a<b\le\infty$, which are called as weighted. Let G be integral operator

$$Gf(x)=\int_x^b G(s,x)f(s)ds \quad (2)$$

with

$$G(s,x)\ge 0 \text{ and } G(s,x)\le G(t,x) \text{ under } s\le t. \quad (3)$$

We denote by $R_p=R_p(J,\rho,v,K)$ a space of measurable functions $f\ge 0$ on J, for which the functional

$$\|f\|_{R_p}\equiv\|\rho f\|_p+\|vGf\|_p \quad (4)$$

is finite, where $\|\cdot\|_p$ is norm of the space L_p, $1<p<\infty$.

From (3) and (4) it follows that

$$\rho^{-1}(\cdot)\equiv\frac{1}{\rho(\cdot)}\in L_{p'}^{loc},\ v(\cdot)\in L_p(a,t),\ v(\cdot)G(\cdot,t)\in L_p(a,t),\ \forall t\in J.$$

Introduce the following relations: we said that $B<<D$ is true, if a constant $c>0$ exists which may depend on p, and inequality $B<cD$ is valid. The relation $B\overset{\cup}{\underset{\cap}{}}D$ means that $B<<D<<B$.

Theorem 1. Let $1<p<\infty$ and G be the operator of type (2) with non-negative kernel $G(s,x)$, and for it condition (3) is satisfied. Then, inequality (1) is valid if and only if $B<\infty$. Here $A\overset{\cup}{\underset{\cap}{}}C$, where C is the least constant in (1).

Keywords: weighted function, integral operator, fractional derivative, operator of fractional differentiation, Hardy's inequality.

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