Some properties of the Schur complement of a block operator matrix

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Abstract: Let T^d be the d-dimensional torus, $L_2(T^d)$ be the Hilbert space of square integrable functions defined on T^d and $L_2^s((T^d)^2)$ be the Hilbert space of square integrable symmetric functions defined on $(T^d)^2$. Denote by H the direct sum of Hilbert spaces $H_1 := L_2(T^d)$ and $H_2 := L_2^s((T^d)^2)$.
In the Hilbert space H, we consider the following block operator matrix

$$A := \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \qquad (1)$$

with the entries $A_{ij} : H_j \to H_i, i \le j, i, j = 1,2$:

$$(A_{11} f_1)(p) = u(p) f_1(p), \ (A_{12} f_2)(p) = \int_{T^d} v(s) f_2(p,s) ds,$$

$$(A_{22} f_2)(p,q) = w(p,q) f_2(p,q), \ f_i \in H_i, \ i = 1,2.$$

Here $u(\cdot)$ and $v(\cdot)$ are real-valued continuous functions on T^d, and $w(\cdot,\cdot)$ is a real-valued continuous symmetric function on $(T^d)^2$.

For a 2×2 block operator matrix (1), there exist two Schur complements, the first of them is formally given by:

$$S_1 : C \setminus \sigma(A_{22}) \to H_1, \ S_1(\lambda) := A_{11} - \lambda - A_{12}(A_{22} - \lambda)^{-1} A_{12}^*, \ \lambda \in \rho(A_{22}).$$

$S_1(\cdot)$ is an analytic operator function defined outside of the spectrum A_{22} and were first used in the theory of matrices [1]. We recall that in the theory of bounded and unbounded block operator matrices, Schur complements are powerful tools to study the spectrum and various spectral properties [2,3].

In this work, the several spectral properties of the first Schur complement $S_1(\lambda)$ related with the number of eigenvalues are studied. In one dimensional case ($d = 1$) for special class of parameter functions of A the existence of the infinitely many eigenvalues of $S_1(\lambda)$ is established. The eigenvalues, it's multiplicities and the corresponding eigenvectors are founded.

Keywords: block operator matrix, Schur complement, eigenvalues, spectrum.

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